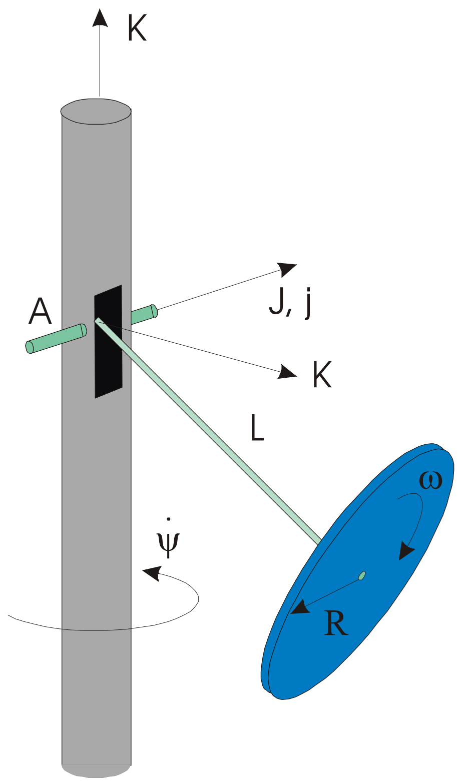
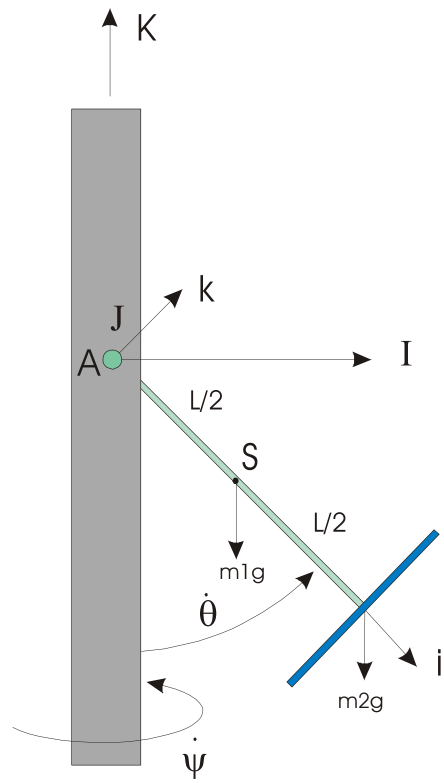
In the following problem, a disk of radius R is attached to a slender shaft of length L, which is pinned to another vertical shaft a point A. The disk and slender shaft swing freely about pin A. The disk rotates about the longitudinal axis of the slender shaft at a constant speed ω. The vertical shaft has a precession rate of $\dot{\psi}$ and the swinging shaft and disk have a nutation rate $\dot{\theta}$ from the vertical shaft. The following will develop the equations of motion for the motion of the system using Lagrange's equations.





Transformations between the fixed frame $\mathcal{A}$ and reference frame $\mathcal{B}$ of the shaft and disk are:

$$\begin{Bmatrix} \bar{I} \\ \bar{J} \\ -\bar{K} \end{Bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & -\sin\theta \end{bmatrix} \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{Bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \\ -\cos\theta & 0 & \sin\theta \end{bmatrix} \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix}$$

Taking the process in reverse yields the transformation from frame $\mathcal{B}$ to $\mathcal{A}$

$$\begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix} = \begin{bmatrix} \sin\theta & 0 & -\cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & \sin\theta \end{bmatrix} \begin{Bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{Bmatrix}$$

The kinetic energy for the disk is the sum of the translational energy and the rotational energy. For the disk, the kinetic energy is

$$T_{disk} = \frac{m_2}{2}(\bar{v}_B \cdot \bar{v}_B) + \frac{1}{2}(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2)$$

First, we calculate the total angular velocity of the disk taking advantage of the coordinate transformations above such that all base vectors are resolved into the frame of the disk

$$\bar{\omega}_{disk} = \dot{\psi}\bar{K} - \dot{\theta}\hat{j} - \omega\hat{i} \Rightarrow -(\dot{\psi}\cos\theta + \omega)\hat{i} - \dot{\theta}\hat{j} + \dot{\psi}\sin\theta\hat{k} \tag{1.1}$$

The linear velocity of the disk is simply the velocity at point B.

$$\bar{v}_B = \bar{v}_A + {}_A\bar{\omega}_B \times L\hat{i} \qquad \text{Where, } {}_A\bar{\omega}_B = -\dot{\psi}\cos\theta\hat{i} - \dot{\theta}\hat{j} + \dot{\psi}\sin\theta\hat{k}$$

Since the velocity at point A is zero, we have

$$\bar{v}_B = {}_A\bar{\omega}_B \times L\hat{i} = [-\dot{\psi}\cos\theta\hat{i} - \dot{\theta}\hat{j} + \dot{\psi}\sin\theta\hat{k}] \times L\hat{i}$$

$$\bar{v}_B = L\dot{\psi}\sin\theta\hat{j} + L\dot{\theta}\hat{k} = L(\dot{\psi}\sin\theta\hat{j} + \dot{\theta}\hat{k}) \tag{1.2}$$

We can now put together the total kinetic energy of the disk

$$T_{disk} = \frac{m_2L^2}{2}[\dot{\psi}^2\sin^2\theta + \dot{\theta}^2] + \frac{1}{2}[I_{xx}(\dot{\psi}\cos\theta + \omega)^2 + I_{yy}\dot{\theta}^2 + I_{yy}\dot{\psi}^2\sin^2\theta] \tag{1.3}$$

Using the principle moments of inertia of the disk

$$I_{xx} = \frac{m_2}{2}R^2 \qquad I_{yy} = I_{zz} = \frac{m_2}{4}R^2$$

Substituting these moments of inertia into equation (1.3) simplifies the kinetic energy for the disk to

$$\begin{aligned} T_{disk} &= \left(\frac{1}{2}\right)m_2L^2\left(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2\right) + \left(\frac{1}{2}\right)\frac{m_2R^2}{4}\left(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2\right) + \left(\frac{1}{2}\right)\frac{m_2R^2}{2}\left(\dot{\psi}\cos\theta + \omega\right)^2 \\ &= \left(\frac{1}{2}\right)\left(m_2L^2 + \frac{m_2R^2}{4}\right)\left(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2\right) + \left(\frac{1}{2}\right)\frac{m_2R^2}{2}\left(\dot{\psi}\cos\theta + \omega\right)^2 \end{aligned} \tag{1.4}$$

Now, we examine the kinetic energy of the shaft. Since the shaft is not rotating about its own longitudinal axis and only rotating about the Z-axis of the vertical shaft and about the pin at point A, the total angular velocity of the shaft is simply

$$\bar{\omega}_{shaft} = {}_A\bar{\omega}_B = \dot{\psi}\bar{K} - \dot{\theta}\hat{j} \Rightarrow -\dot{\psi}\cos\theta\hat{i} - \dot{\theta}\hat{j} + \dot{\psi}\sin\theta\hat{k} \tag{1.5}$$

The linear velocity of the mass center of the shaft is

$$v_s = v_A + \bar{\omega}_{shaft} \times (L/2)\hat{i}$$

$$v_s = [-\dot{\psi}\cos\theta\hat{i} - \dot{\theta}\hat{j} + \dot{\psi}\sin\theta\hat{k}] \times (L/2)\hat{i} = (L/2)\dot{\psi}\sin\theta\hat{j} + (L/2)\dot{\theta}\hat{k}$$

$$v_s = (L/2)[\dot{\psi}\sin\theta\hat{j} + \dot{\theta}\hat{k}] \tag{1.6}$$

Now having the total angular velocity of the shaft and its total linear velocity, we can now obtain the total kinetic energy of the shaft

$$T_{shaft} = \frac{m_1}{2}(\bar{v}_s \cdot \bar{v}_s) + \frac{1}{2}\left[J_{xx}\omega_{shaft-x}^2 + J_{yy}\omega_{shaft-y}^2 + J_{zz}\omega_{shaft-z}^2\right]$$

Assuming a slender rod, the moments of inertia of the rod for rotation in frame $\mathcal{B}$ are

$$J_{xx} = 0 \qquad J_{yy} = J_{zz} = \frac{m_1}{12} L^2$$

Collecting all the terms and making the substitutions, the kinetic energy of the shaft simplifies to

$$T_{shaft} = \frac{m_1 L^2}{6}\left[\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta\right] \tag{1.7}$$

Summing the energy of the shaft and disk, the total kinetic energy of the system can now be written as

$$\begin{aligned} T_{total} &= \left(\left(\frac{m_2 R^2}{4} + m_2 L^2\right)\left(\frac{1}{2}\right)\left[\dot{\psi}^2 \sin^2\theta + \dot{\theta}^2\right]\right) + \left(\frac{m_2 R^2}{2}\right)\left(\frac{1}{2}\right)\left(\dot{\psi}\cos\theta + \omega\right)^2 \\ &\quad + \frac{m_1 L^2}{3}\left(\frac{1}{2}\right)\left[\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta\right] \\ &= \left(\frac{m_2 R^2}{4} + m_2 L^2 + \frac{m_1 L^2}{3}\right)\left(\frac{1}{2}\right)\left[\dot{\psi}^2 \sin^2\theta + \dot{\theta}^2\right] + \left(\frac{m_2 R^2}{2}\right)\left(\frac{1}{2}\right)\left(\dot{\psi}\cos\theta + \omega\right)^2 \end{aligned} \tag{1.8}$$

In order to simplify the equations we make the following substitutions

$$I' = \left(\frac{m_2 R^2}{2}\right) \text{ and } I'' = \left(\frac{m_2 R^2}{4} + m_2 L^2 + \frac{m_1 L^2}{3}\right)$$

Equation (1.8) can now be written as

$$T_{total} = I''\left(\frac{1}{2}\right)\left[\dot{\psi}^2 \sin^2\theta + \dot{\theta}^2\right] + I'\left(\frac{1}{2}\right)\left(\dot{\psi}\cos\theta + \omega\right)^2 \tag{1.9}$$

Now, we need to derive the total potential energy of the system. Using point A as the datum, the potential energy is simply the sum of individual potentials of the shaft and disk

$$U = -m_2 g L\cos\theta - \frac{m_1 g L\cos\theta}{2} = -gL\cos\theta\left(\frac{m_1}{2} + m_2\right) \tag{1.10}$$

We now have everything needed to form the Lagrangian Function, *L.* Using the Lagrangian we can systematically write the final equations of motion for each generalized coordinate.

$$L = T - U$$

$$L = I''\left(\frac{1}{2}\right)\left[\dot{\psi}^2 \sin^2\theta + \dot{\theta}^2\right] + I'\left(\frac{1}{2}\right)\left(\dot{\psi}\cos\theta + \omega\right)^2 + gL\cos\theta\left(\frac{m_1}{2} + m_2\right) \tag{1.11}$$

First, let's start with $\dot{\psi}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) = \frac{d}{dt}\left[I''\dot{\psi}^2 \sin^2\theta + I'\left(\dot{\psi}\cos\theta + \omega\right)\cos\theta\right]$$
$$= I''\left(\ddot{\psi}\sin^2\theta + \dot{\psi}2\sin\theta\cos\theta\dot{\theta}\right) + I'\cos\theta\left(\ddot{\psi}\cos\theta - \dot{\psi}\sin\theta\dot{\theta}\right)$$
$$-I'\sin\theta\dot{\theta}\left(\dot{\psi}\cos\theta + \omega\right) \quad (1.12)$$
$$= \ddot{\psi}\left(I''\sin^2\theta + I'\cos^2\theta\right) + \dot{\psi}\left(I''\sin 2\theta\dot{\theta} - I'\sin 2\theta\dot{\theta}\right) - I'\sin\theta\dot{\theta}\omega$$
$$= \ddot{\psi}\left(I''\sin^2\theta + I'\cos^2\theta\right) + \dot{\psi}\sin 2\theta\dot{\theta}\left(I'' - I'\right) - I'\sin\theta\dot{\theta}\omega$$

Since there are no terms that are functions of ψ, $\left(\frac{\partial L}{\partial \psi}\right) = 0$.

For the θ coordinate we have the following

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = I''\ddot{\theta} \quad (1.13)$$

$$\left(\frac{\partial L}{\partial \theta}\right) = I''\left(\dot{\psi}\sin\theta\cos\theta\right) - I'\left(\dot{\psi}\cos\theta + \omega\right)\dot{\psi}\sin\theta - gL\sin\theta\left(\frac{m_1}{2} + m_2\right)$$
$$= \left(I'' - I'\right)\dot{\psi}^2\sin\theta\cos\theta - I'\dot{\psi}\omega\sin\theta + gL\sin\theta\left(\frac{m_1}{2} + m_2\right) \quad (1.14)$$

Since there are no non-conservative forces acting on the system, the virtual work done in the $\delta\psi$ and $\delta\theta$ directions are zero

$$Q_\psi = Q_\theta \equiv 0$$

We now have everything we need to assemble the equations of motion. The corresponding Lagrangian equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = 0$$

for the ψ and θ coordinates we have

$$\ddot{\psi}\left(I''\sin^2\theta + I'\cos^2\theta\right) + \dot{\psi}\sin 2\theta\dot{\theta}\left(I'' - I'\right) - I'\sin\theta\dot{\theta}\omega = 0 \quad (1.15)$$

$$I''\ddot{\theta} + \left(I' - I''\right)\dot{\psi}^2\sin\theta\cos\theta + I'\dot{\psi}\omega\sin\theta + gL\sin\theta\left(\frac{m_1}{2} + m_2\right) = 0 \quad (1.16)$$