Generalized Coordinates and Degrees of Freedom

In deriving the equations of motion for a system, one must begin with selecting a set of coordinates for the problem. A set of coordinates or parameters, which **uniquely** describes the geometric position and/or orientation of body, or system of bodies, is called the *generalized coordinates* of the system. The *minimum* number of independent generalized coordinates necessary to completely describe the geometry is known as the *number of degrees of freedom*. In general, a set of generalized coordinates for a given system is not unique. Any set of coordinates that can uniquely describe the geometry of the system is a suitable set of coordinates. This doesn't mean, however, that any set of coordinates should be used. As you can guess, for given problem, based on the geometry and type of information being examined, some sets of coordinates will allow for an easier solution with more physical insight than others. For example (Figure 1), one may solve the simple planar motion of swinging pendulum using Cartesian coordinates in x and y but the coordinate system that provides the clearest view of the motion is obviously polar coordinates. Therefore, even though we may appropriately select x and y as the set of generalized coordinates, the most convenient set of coordinates is the angle θ.

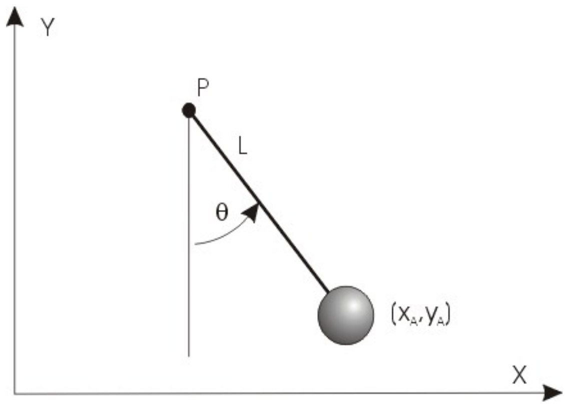

Figure 1

Below in (Figure 2), a bar lies in the x-y plane. The bar is unconstrained and can be placed in any position or orientation. Given the two end points A and B, four generalized coordinates, (X_A, Y_A) and (X_B, Y_B) are sufficient to completely describe the position and orientation of the bar. However, it is to see that only three coordinates are really needed. If we know the position at one end of the bar and the bar's rotation angle, θ, then we can completely describe the geometry of the bar. Therefore, the total number of degrees of freedom for the bar is three. As previously stated, it is the *minimum* number of generalized coordinates that determines the number of degrees of freedom and each coordinate must be independent. Here, the four coordinates, (X_A, Y_A)

and $\left(X_B, Y_B\right)$, are not independent. We can quickly see that by specifying any 3, the 4th is determined.

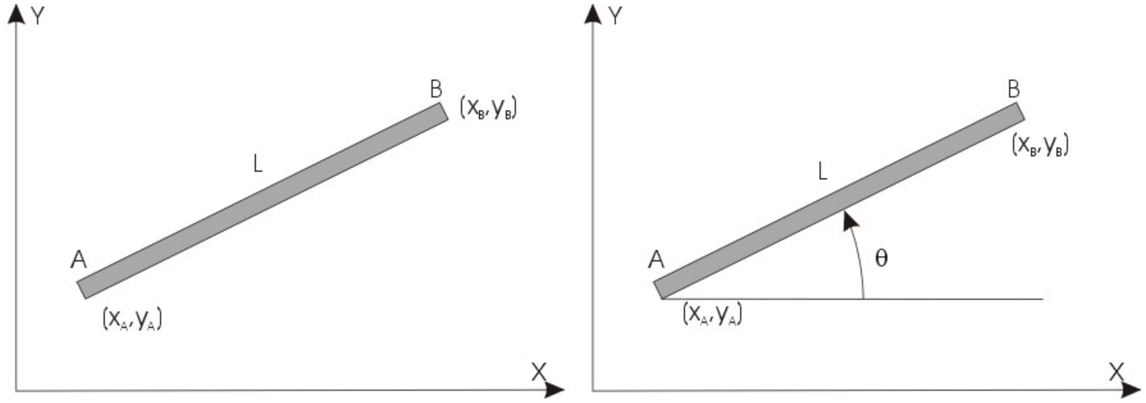

Figure 2

For systems that can have a variety of sets of generalized coordinates, it will always be possible to map one set of coordinates to another. In the above case, the mapping of the 4 generalized coordinates $\left(X_A, Y_A, X_B, Y_B\right)$ to coordinates in $\left(X_A, Y_A, \theta\right)$ is the accomplished by the following relationship

$$X_B = X_A + L\cos\theta \text{ and } Y_B = Y_A + L\sin\theta$$

These equations clearly illustrate that the 4 generalized coordinates $\left(X_A, Y_A, X_B, Y_B\right)$ are not independent and are related by the angle θ.

Now, let's pin one end of the bar at point A.

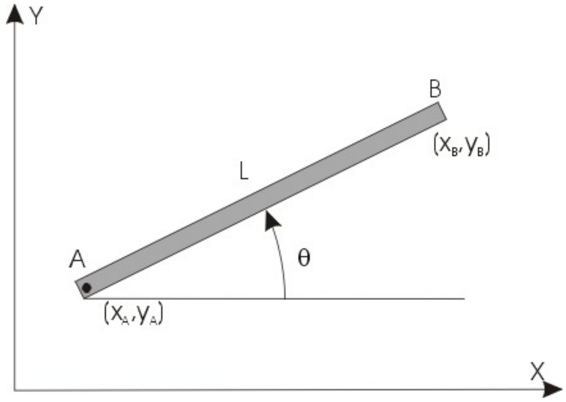

Figure 3

By pinning the end of the bar (Figure 3) such that X_A and Y_A are constrained, the number of degrees of freedom are reduced by two. Now, the only coordinate necessary to describe the geometry of the system is the rotation angle θ.

If we continued and pinned the other end of the bar at point B (Figure 4), then the system would be completely constrained, thus having zero degrees of freedom. Note, instead of pinning the bar at point B, we could have just as easily fixed the angle θ to completely constrain the bar.

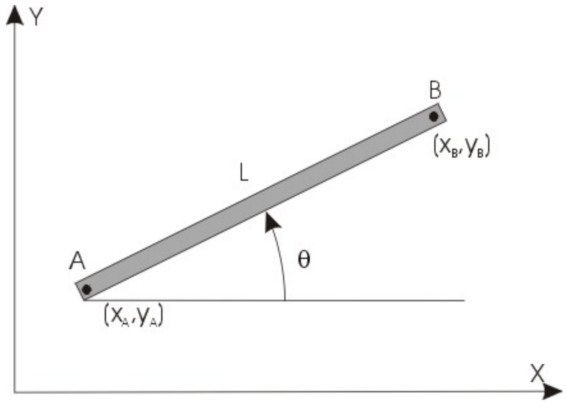

Figure 4

Generally, the constraint of a pin results reduces the number of degrees of freedom by two. In this instance, however, pinning the bar at B only reduced the system by one – for two reasons. First, you obviously cannot reduce the number of degree of freedom by more than it has. And second, the dependent relationship between the rotation angle and coordinates at B (given that there is already a pin at A) means that endpoint B really only represents a single degree of freedom, namely θ. Thus constraining B only reduced the number of degrees of freedom by one.

Now let's take a final look at a multi-body system. Below in Figure 5, two masses are connected by a rods. The first mass, $m_{1,}$ is suspended by a rod which is also connected to a ball and socket joint at P. The second mass, $m_{2,}$ is connected to the first mass suspended by an extensible rod and a ball and socket joint at $m_{1.}$ Before selecting the generalized coordinates, let's first determine the number of degrees of freedom.

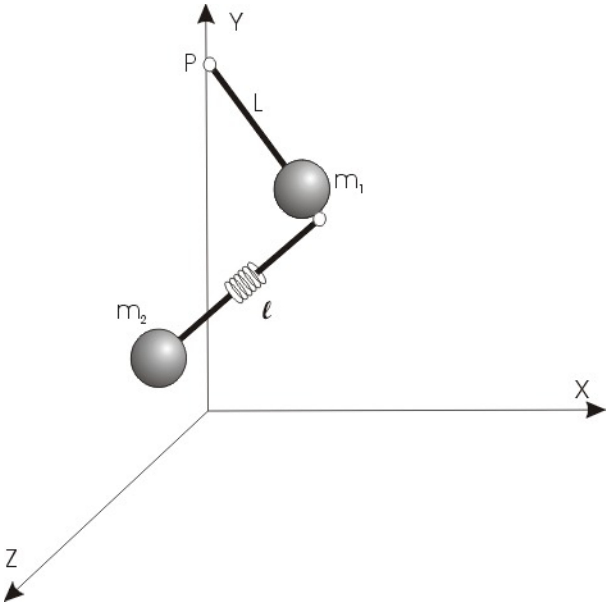

Figure 5

For the first mass, m_1 is connected to an **inextensible** rod which is free to rotate in 3 directions. It can swing through a nutation angle and precession angle and but can also rotate about the long axis of the rod. The *x,y,z* position of m_1 is constrained by the rod and the ball and socket joint such that m_1 will always be a distance L from P. Since its position is constrained but it has the freedom to rotate in 3 directions, m_1 contributes 3 degrees of freedom to the complete system.

The situation for the second mass, m_2, is almost identical to that of $m_{1.}$ The second mass is attached to the first via a rod and ball and socket joint but in this case the rod is **extensible**. The extensibility of the rod adds an additional degree of freedom. M_2's position is still constrained by the length of the rod, l, but l is now allowed to change. The extensible rod and the ball and socket joint give m_2 a total of 4 degrees of freedom. Summing these with those of m_1 we now have a total of 7 degrees of freedom for the entire system.

Now that we have a clear understanding of the number of degrees of freedom and their origin, selecting the generalized coordinates becomes a much easier task. It would only make sense to select 3 rotation angles for m_1, 3 rotation angles for m_2, and one additional coordinate for the extensibility of the second rod, l. Therefore we select for the system the generalized coordinates $(\phi_1, \theta_1, \psi_1, \phi_2, \theta_2, \psi_2, l)$.