

Finite Rotations

In kinematics, there are two commonly used sets of transformations for describing the change in orientation of a body in space or the rotation of a set of reference axes with respect to another. These transformations are categorized as either *space fixed* and *body fixed* rotations. In the first case, each rotation is about an axis in which remains fixed in space with respect to the body or axes being rotated. Regardless of the change in orientation of a body or axes, space fixed rotations always occur about the same set of axes. Body fixed rotations, however, are rotations that occur about axes fixed in the reference of the body. Thus, the absolute orientation of each axis depends on the previous rotation. In both space fixed and body fixed rotations, the final orientation of a body or set of reference axes depends on both the sequence of the rotations and their magnitude.

Although *space fixed* and *body fixed* rotations generally provide different results, the matrix transformations used to describe the rotations are the same. What differs is the order or sequence in which they are applied. The following matrices are used to describe rotations about each axis in a Cartesian coordinate system

$$\begin{aligned}
 [T_x] &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix} \\
 [T_y] &= \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \\
 [T_z] &= \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}
 \end{aligned} \tag{1.1}$$

An obvious characteristic of the above matrices is that they are orthogonal – or better yet orthonormal. This means that the determinate of the matrix is unity and that the magnitude of any vector transformed by these matrices remains the same (i.e. no scaling). Also, orthogonal/orthonormal matrices have the characteristic that their transpose is equal to their inverse. In short

$$[T]^T = [T]^{-1}$$

There are two common uses of the above transformations. The first is to be able to describe a set of basis vector in one frame of reference in terms of another. In any problem, just like working with a consistent set of units, one must make sure there is consistent set of basis vectors used in the formulation of the problem.

The second common use for the above transformations is transforming a point or body in a reference system to a new orientation. These two uses are very closely related in that the former can be viewed as transformation of axes relative to a fixed point or frame and then latter viewed a transformation of a point or frame relative to a fixed set of axes. The transformations are nearly identical except for a change in sign for the rotation angles.

The following is an example of using the transformations for a change in coordinate system. Given two coincident frames A and B , let's rotate a reference frame B , with basis vector $\hat{i}_B, \hat{j}_B, \hat{k}_B$, about the \bar{K}_A axis in frame A , which has basis vectors $\bar{I}_A, \bar{J}_A, \bar{K}_A$. The transformation of $\bar{I}_A, \bar{J}_A, \bar{K}_A$ to obtain $\hat{i}_B, \hat{j}_B, \hat{k}_B$ is accomplished with the following matrix operation.

$$\hat{u}_B = \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}_B = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix}_A = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \hat{u}_A$$

Expanding this gives us the relationship between of the basis vectors $\hat{i}_B, \hat{j}_B, \hat{k}_B$ in frame B to the basis vectors $\bar{I}_A, \bar{J}_A, \bar{K}_A$ in frame A . The opposite transformation (applying a negative value for the rotation angle) gives us the transformation of the $\hat{i}_B, \hat{j}_B, \hat{k}_B$ basis vectors in frame B in terms of the basis $\bar{I}_A, \bar{J}_A, \bar{K}_A$ in frame A . Together we have

$$\begin{aligned} \hat{i}_B &= \cos \psi \bar{I}_A + \sin \psi \bar{J}_A & \bar{I}_A &= \cos \psi \hat{i}_B - \sin \psi \hat{j}_B \\ \hat{j}_B &= -\sin \psi \bar{I}_A + \cos \psi \bar{J}_A & \bar{J}_A &= \sin \psi \hat{i}_B + \cos \psi \hat{j}_B \\ \hat{k}_B &= \bar{K}_A & \bar{K}_A &= \hat{k}_B \end{aligned} \quad \text{and}$$

This process is repeated as many times as necessary for a given problem depending on the number of frames of reference used and the final reference frame chosen for completing the solution.

Addressing the second most common use of these transformations we want to look at the transformation of axes and points and see also see how they are related.

Again, let's assume we have a fixed reference system A and reference system B which we will rotate 45° about the \bar{K}_A axis. Let's also have a point $\bar{p}_A = 2\bar{I}_A + 3\bar{J}_A$. First, we will rotate the B axes to yield the point relative to reference frame B . Similarly to previous example, the transformation of basis vector between frame A and B is

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}_B = \begin{bmatrix} \cos 45^\circ & \sin 45^\circ & 0 \\ -\sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix}_A$$

$$\begin{aligned} \hat{i}_B &= \cos 45^\circ \bar{I}_A + \sin 45^\circ \bar{J}_A = 0.707\bar{I}_A + 0.707\bar{J}_A \\ \hat{j}_B &= -\sin 45^\circ \bar{I}_A + \cos 45^\circ \bar{J}_A = -0.707\bar{I}_A + 0.707\bar{J}_A \\ \hat{k}_B &= \bar{K}_A \end{aligned} \quad (1.2)$$

the reverse transformation yields

$$\begin{aligned} \bar{I}_A &= \cos 45^\circ \hat{i}_B - \sin 45^\circ \hat{j}_B = 0.707\hat{i}_B - 0.707\hat{j}_B \\ \bar{J}_A &= \sin 45^\circ \hat{i}_B + \cos 45^\circ \hat{j}_B = 0.707\hat{i}_B + 0.707\hat{j}_B \\ \bar{K}_A &= \hat{k}_B \end{aligned} \quad (1.3)$$

To obtain the point \bar{p}_A in the B reference frame all we need to do is substitute the $\bar{I}_A, \bar{J}_A, \bar{K}_A$ basis vector using the relationship in (1.3).

$$\begin{aligned} \bar{p}_A &= 2\bar{I}_A + 3\bar{J}_A \\ \bar{p}_B &= 2(0.707\hat{i}_B - 0.707\hat{j}_B) + 3(0.707\hat{i}_B + 0.707\hat{j}_B) \\ &= 3.536\hat{i}_B + 0.707\hat{j}_B \end{aligned}$$

From this result it is obvious that the transformation is a transformation of the axes of frame B relative to the stationary point \bar{p} in frame A

In many applications, it is desired to actually transform or rotate the point instead of the axes. In this instance, let's perform the same transformation above but this time we'll rotate the point instead of the axes. To perform this operation, we will fix point \bar{p} in frame B prior to the rotation about \bar{K}_A such that \bar{p} now becomes \bar{p}_B and

remains stationary relative to frame B. Again, substituting the transformations of the basis vectors in (1.2) yields \bar{p}_A .

$$\begin{aligned}\bar{p}_B &= 2\hat{i}_B + 3\hat{j}_B \\ \bar{p}_A &= 2(0.707\bar{I}_A + 0.707\bar{J}_A) + 3(-0.707\bar{I}_A + 0.707\bar{J}_A) \\ &= -0.707\bar{I}_A + 3.536\bar{J}_A\end{aligned}$$

In general, it's all a matter of perspective. In this case we wanted to rotate point \bar{p} but what we actually did was rotate frame A 45° in the opposite direction and obtain the new position of the point relative to frame A. In short, to rotate the coordinate system, use the matrices in (1.1) with positive values for the angles. If it is desired to obtain the transformation of a point, then use equations (1.1) rotating the axes in the opposite direction by substituting negative values for the angles.

Body Fixed Rotations

As mentioned, body fixed rotations are rotations which occur about the axes fixed a body frame of reference. In other word, regardless of a change in position or orientation, the reference axis system moves with the body and all rotations are about the axes in body. For example, a body undergoes a sequence of rotations ϕ_x , θ_y , ψ_z about the x, y, and z body axes, respectively. The first rotation, ϕ_x , is about the x-axis in the body frame's coordinate system. Now with the body in its new orientation, the second rotation, θ_y , occurs about the y-axis fixed in the body.

Finally, the third rotation ψ_z occurs about the body's z-axis. Although the absolute orientation of the axes have changed relative to the fixed reference system for each rotation, the axes about which the body rotates remain fixed when viewed from the body's frame of reference.

For a sequence of body fixed rotations, the matrix transformations are combined in the order in which the rotations occur with each successive rotation pre-multiplying the previous. A final transformation matrix of n body fixed rotations has the form

$$[T] = [T_n] \dots [T_3][T_2][T_1] \tag{1.4}$$

For the above example, a final transformation for the sequence of rotations ϕ_x , θ_y , and ψ_z , would be

$$[T] = [T_\psi][T_\theta][T_\phi]$$

It should be noted that, in general for a Cartesian coordinate systems, the sequence of rotations most commonly used occur about the z-axis first, then the y-axis, then

the x-axis. As you'll see in the next section, reversing the order correspond to a completely different set of rotations with a completely different intent.

Space Fix Rotations

Space fixed rotations are very similar to body fixed rotation with the exceptions noted above. Each rotation occurs about a fixed set of axes regardless of the orientation of the body. Like body fixed rotations, a sequence of space fixed rotations can be combined into a single transformation; however, the order in which the transformations are applied is in reverse. For a sequence of n space fixed rotations, a final matrix $[T]$ has the following form.

$$[T] = [T_1][T_2][T_3] \dots [T_n] \quad (1.5)$$

The similarities between equations (1.4) and (1.5) can be both a source of convenience and a significant source for errors and care should be taken to make sure the proper sequence is used for the desired set of transformations.