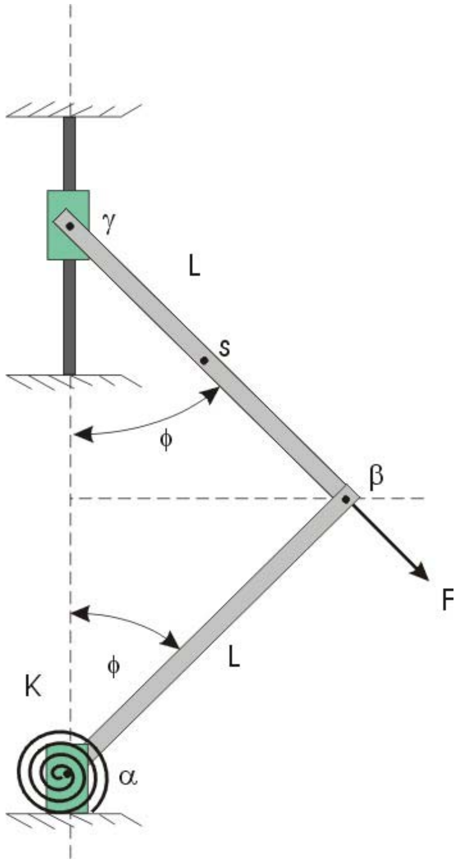



In the figure, two bars are hinged in the vertical position. The lower bar is pinned to a fixed support and its rotation impeded by a torsional spring. The spring is in a relaxed position when $\phi = \phi_0$. The upper bar is pinned to a collar which is constrained to move along a vertical shaft. A force, F, is applied at the point β, where a pin connects the two bars. The force remains parallel to the upper bar as the two bars rotate. Write the equations of motion using Lagrange's equations.

The first objective is to compute the virtual work performed by the force F. Here, F does work through the point β as the bars rotate through the rotation angle ϕ. (In this problem ϕ will be our generalized coordinate). By definition, the virtual work is

$$\delta W = \overline{F} \cdot \delta \overline{r}_\beta$$

Where F is the generalized force and $\delta \overline{r}_\beta$ is the change in the position vector of the point β. For $\overline{r}_\beta$ we have

$$\overline{r}_\beta = L \sin \phi \hat{i} + L \cos \phi \hat{j} \qquad (1.1)$$

To compute $\delta \overline{r}_\beta$, we allow the bars to rotate by an incremental amount $\delta\phi$. The incremental change in the position of the pin β is then

$$\begin{aligned} \delta \overline{r}_\beta &= \frac{\partial \overline{r}_\beta}{\partial \phi} \delta\phi \\ &= L(\cos\phi \hat{i} - \sin\phi \hat{j})\delta\phi \end{aligned} \qquad (1.2)$$

The force F in vector form is $F = \left( F \sin \phi \hat{i} - F \cos \phi \hat{j} \right)$. Applying equation (1.2) we're now able compute the virtual work

$$\begin{aligned} \delta W &= (F \sin\phi \hat{i} - F \cos\phi \hat{j}) \cdot (L \cos\phi \hat{i} - L \sin\phi \hat{j})\delta\phi \\ &= 2FL \sin\phi \cos\phi \delta\phi \\ &= FL \sin 2\phi \delta\phi \end{aligned} \qquad (1.3)$$

Having the virtual work now allows us to compute the generalized force. Remember, the generalized force is what turns up in the equations of motion – not the virtual work. The generalized force $Q_\phi$ for the coordinate $\phi$ is simply

$$Q_\phi = \frac{\delta W}{\delta \phi} = FL\sin 2\phi \tag{1.4}$$

To compute the kinetic energy we will need to calculate the translational velocity of the upper bar at its center of mass. We'll first start by calculating the velocity of the pin at $\beta$.

$$\begin{aligned} \bar{r}_\beta &= L\left(\sin\phi\hat{i} + \cos\phi\hat{j}\right) \\ \bar{v}_\beta &= L\left(\dot{\phi}\cos\phi\hat{i} - \dot{\phi}\sin\phi\hat{j}\right) \\ &= L\dot{\phi}\left(\cos\phi\hat{i} - \sin\phi\hat{j}\right) \end{aligned} \tag{1.5}$$

The velocity of the center of mass for the upper bar is then

$$\begin{aligned} \bar{v}_s &= \bar{v}_\beta + \dot{\phi}\hat{k} \times \left(-\frac{L}{2}\sin\phi\hat{i} + \frac{L}{2}\cos\phi\hat{j}\right) \\ &= L\dot{\phi}\left(\cos\phi\hat{i} - \sin\phi\hat{j}\right) - \frac{L}{2}\dot{\phi}\cos\phi\hat{i} - \frac{L}{2}\dot{\phi}\sin\phi\hat{j} \\ &= \frac{L}{2}\dot{\phi}\cos\phi\hat{i} - \frac{3L}{2}\dot{\phi}\sin\phi\hat{j} \end{aligned} \tag{1.6}$$

We should note that we do not need to calculate the velocity of the lower bar since the lower bar is rotating about the fixed point $\alpha$. The bar is in pure rotation and only the rotational term shows up in its kinetic energy. The upper bar has both translational and rotational motion and is therefore in general motion. Subsequently, the upper bar's kinetic energy will have two terms. Writing the total kinetic energy for the system we have

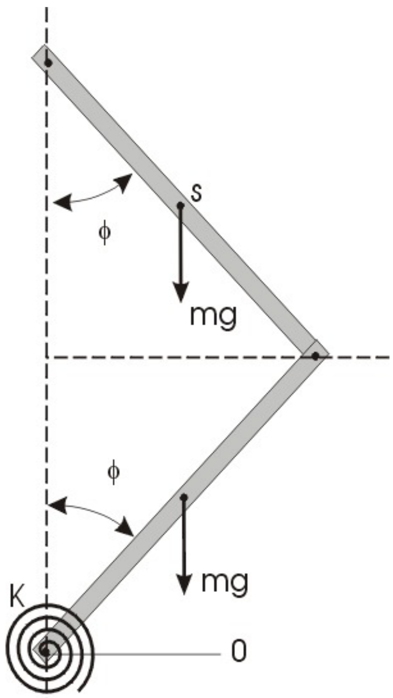
$$\begin{aligned} T &= \frac{1}{2}H_a\omega + \frac{1}{2}m\bar{v}_s \cdot \bar{v}_s + \frac{1}{2}H_s \cdot \omega \\ &= \frac{1}{2}I_\alpha\dot{\phi}^2 + \frac{1}{2}m\left(\frac{L}{2}\dot{\phi}\cos\phi\hat{i} - \frac{3L}{2}\dot{\phi}\sin\phi\hat{j}\right)\cdot\left(\frac{L}{2}\dot{\phi}\cos\phi\hat{i} - \frac{3L}{2}\dot{\phi}\sin\phi\hat{j}\right) + \frac{1}{2}I_s\dot{\phi}^2 \\ &= \frac{1}{2}I_\alpha\dot{\phi}^2 + \frac{1}{2}m\dot{\phi}^2\left(\frac{L}{2}\right)^2\left(\cos^2\phi + 9\sin^2\phi\right) + \frac{1}{2}I_s\dot{\phi}^2 \end{aligned} \tag{1.7}$$

The moments of inertia $I_\alpha$ and $I_s$ are moments of the lower bar about the pin α and the upper bar its center of mass.

$$I_\alpha = \frac{mL^2}{3} \qquad \text{and} \qquad I_s = \frac{mL^2}{12}$$

Making these substitutions we have the final form for the kinetic energy

$$\begin{aligned} T &= \frac{1}{2}\frac{mL^2}{3}\dot{\phi}^2 + \frac{1}{2}m\left(\dot{\phi}\frac{L}{2}\right)^2\left(\cos^2\phi + 9\sin^2\phi\right) + \frac{1}{2}\frac{mL^2}{12}\dot{\phi}^2 \\ &= \frac{1}{2}\dot{\phi}^2 mL^2\left[\frac{5}{12} + \frac{1}{4}\left(\cos^2\phi + 9\sin^2\phi\right)\right] \qquad (1.8) \\ &= \dot{\phi}^2 mL^2\left(\frac{1}{3} + \sin^2\phi\right) \end{aligned}$$




The next step is to form the potential energy which is a sum of potentials in both upper and lower bars and the torsional energy stored in the spring. Using the pin at α as the zero reference line we have for the potential energy for each bar

$$U_l = mg\frac{L}{2}\cos\phi \qquad \text{and} \qquad U_u = mgL(\cos\phi + \frac{1}{2}\cos\phi)$$

The torsional energy stored in the spring is the change in $\phi$ from its rest position $\phi_o$.

$$U_k = \frac{1}{2}K\left(\phi - \phi_o\right)^2$$

Summing these energies we have for the total potential energy for the system

$$\begin{aligned} U &= mg\frac{L}{2}\cos\phi + mgL\left(\cos\phi + \frac{1}{2}\cos\phi\right) + \frac{1}{2}K\left(\phi - \phi_o\right)^2 \qquad (1.9) \\ &= 2mgL\cos\phi + \frac{1}{2}K\left(\phi - \phi_o\right)^2 \end{aligned}$$

Having both the total kinetic and total potential energies we are ready to write the Lagrangian function $L = T - U$. Using equations (1.8) and (1.9) gives us the following

$$L = mL^2\dot{\phi}^2\left(\frac{1}{3} + \sin^2\phi\right) - 2mgL\cos\phi - \frac{1}{2}K\left(\phi - \phi_o\right)^2 \qquad (1.10)$$

The equations of motion for the generalized coordinate $\phi$ will be

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)-\frac{\partial L}{\partial \phi}=Q_{\phi} \qquad (1.11)$$

Starting with the first term,

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)&=\frac{d}{dt}\left[2mL^2\dot{\phi}\left(\frac{1}{3}+\sin^2\phi\right)\right]\\&=2mL^2\ddot{\phi}\left(\frac{1}{3}+\sin^2\phi\right)+2mL^2\dot{\phi}^2 2\sin\phi\cos\phi\\&=2mL^2\left[\ddot{\phi}\left(\frac{1}{3}+\sin^2\phi\right)+\dot{\phi}^2\sin 2\phi\right]\end{aligned} \qquad (1.12)$$

The second term in (1.11) will be

$$\begin{aligned}\frac{\partial L}{\partial \phi}&=mL^2\dot{\phi}^2 2\sin\phi\cos\phi+2mgL\sin\phi-k\left(\phi-\phi_0\right)\\&=mL^2\dot{\phi}^2\sin 2\phi+2mgL\sin\phi-k\left(\phi-\phi_0\right)\end{aligned} \qquad (1.13)$$

Substituting the results from equations (1.12), (1.13) and (1.4) into (1.11), we have the following for the final equations of motion for the system.

$$\begin{aligned}&mL^2\left[\ddot{\phi}\left(\frac{1}{3}+\sin^2\phi\right)+\dot{\phi}^2\sin 2\phi\right]-mL\dot{\phi}^2\sin 2\phi-2mgL\sin\phi+k\left(\phi-\phi_0\right)=FL\sin 2\phi\\&2mL^2\left[\ddot{\phi}\left(\frac{1}{3}+\sin^2\phi\right)+\frac{\dot{\phi}^2}{2}\sin 2\phi\right]-2mgL\sin\phi+k\left(\phi-\phi_0\right)=FL\sin 2\phi\end{aligned} \qquad (1.14)$$