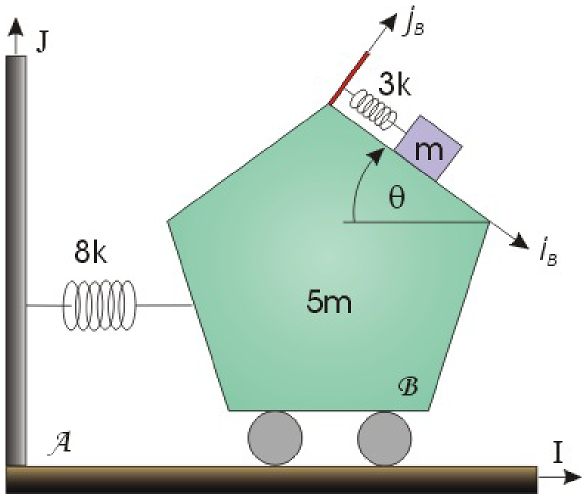

A cart, of mass *5m*, rolls along the ground and is connected to a wall by a linear spring with a coefficient of *8K*. From the top of the cart a block, of mass *m*, is suspended down the front incline of the cart by another spring with a spring coefficient of *3K*. The cart rolls along the ground without slipping. Derive the equations of motion using Lagrange's equations.

Solution:
The first thing to look at in this problem is the selection of the generalized coordinates. We can immediately see that we will require two frames of reference. The first frame, $\mathcal{A}$, is the inertial frame in which the cart rolls along the ground. The second frame, $\mathcal{B}$, will reside in the frame of the cart oriented along the forward incline. The suspended block at the top of the cart moves with respect to this frame. Our two generalized coordinates are:

1. x, in the $\overline{I}_A$ of the cart along the ground
2. μ in the $\hat{i}_B$ direction for the motion of the block along the forward incline

The first thing we will need is a set of transformation equations for changing basis vectors between frames $\mathcal{A}$ and $\mathcal{B}$.

$$\begin{aligned} \hat{i}_B &= \cos\theta \overline{I}_A + \sin\theta \overline{J}_A \\ \hat{j}_B &= -\sin\theta \overline{I}_A + \cos\theta \overline{J}_A \\ \hat{k}_B &= \overline{K}_A \end{aligned} \tag{1.1}$$

We'll first start with writing the kinetic energy of the cart. As the cart move along the ground in the $\overline{I}_A$ direction its kinetic energy is

$$T_{cart} = \frac{1}{2}(5m)\dot{x}^2 \tag{1.2}$$

Next, we need the kinetic energy of the block. The total linear velocity of the block is the sum of the velocity of the cart plus the velocity of the block with respect to the cart. For the kinetic energy of the block we have

$$T_{block} = \frac{1}{2} m\left[\left(\dot{x}\bar{I}_A + \dot{\mu}\hat{i}_B\right)\cdot\left(\dot{x}\bar{I}_A + \dot{\mu}\hat{i}_B\right)\right] \tag{1.3}$$

Using the transformations in equation (1.1) we write the kinetic energy of the block as

$$\begin{aligned} T_{block} &= \frac{m}{2}\left[\left(\dot{x}\bar{I}_A + \dot{\mu}\left(\cos\theta\bar{I}_A + \sin\theta\bar{J}_A\right)\right)\cdot\left(\dot{x}\bar{I}_A + \dot{\mu}\left(\cos\theta\bar{I}_A + \sin\theta\bar{J}_A\right)\right)\right] \\ &= \frac{m}{2}\left[\left(\left(\dot{x} + \dot{\mu}\cos\theta\right)\bar{I}_A + \dot{\mu}\sin\theta\bar{J}_A\right)\cdot\left(\left(\dot{x} + \dot{\mu}\cos\theta\right)\bar{I}_A + \dot{\mu}\sin\theta\bar{J}_A\right)\right] \\ &= \frac{m}{2}\left[\left(\dot{x} + \dot{\mu}\cos\theta\right)^2 + \dot{\mu}^2\sin^2\theta\right] \end{aligned} \tag{1.4}$$

The total kinetic energy for the system is the sum of equations (1.2) and (1.4)

$$T = \frac{5m}{2}\dot{x}^2 + \frac{m}{2}\left[\left(\dot{x} + \dot{\mu}\cos\theta\right)^2 + \dot{\mu}^2\sin^2\theta\right] \tag{1.5}$$

The next step is to determine the total potential energy for the system. In this problem, the potential will consist of the gravitational potential for the block and the conservative forces in the springs. Since the cart does not move in the vertical direction, there is no change in its potential.

Using the top of the cart as the zero reference datum, we have for the potential energy of the block

$$U_{block} = -mg\mu\sin\theta \tag{1.6}$$

If we let the undeflected spring positions for the generalized coordinates be zero, we have for the potential energy stored in the springs the following equation

$$U_{springs} = \frac{1}{2}(8k)x^2 + \frac{1}{2}(3k)\mu^2 \tag{1.7}$$

The total potential for the system is then

$$U = -mg\mu\sin\theta + \frac{1}{2}(8k)x^2 + \frac{1}{2}(3k)\mu^2 \tag{1.8}$$

Having both the total kinetic and potential energy of the system, we can now write the Lagrangian, *L=T-U*.

$$L = \frac{5m}{2}\dot{x}^2 + \frac{m}{2}\left[\left(\dot{x} + \dot{\mu}\cos\theta\right)^2 + \dot{\mu}^2\sin^2\theta\right] + mg\mu\sin\theta - \frac{1}{2}\left(8k\right)x^2 - \frac{1}{2}\left(3k\right)\mu^2 \quad (1.9)$$

Let's start with taking derivatives for the x coordinate

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{d}{dt}\left[5m\dot{x} + m\left(\dot{x} + \dot{\mu}\cos\theta\right)\right] \\ &= 6m\ddot{x} + m\ddot{\mu}\cos\theta\end{aligned} \quad (1.10)$$

$$\frac{\partial L}{\partial x} = -8kx \quad (1.11)$$

For the μ coordinate we have the following derivatives

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mu}}\right) &= \frac{d}{dt}\left[m\left(\left(\dot{x} + \dot{\mu}\cos\theta\right)\cos\theta + \dot{\mu}\sin^2\theta\right)\right] \\ &= \frac{d}{dt}\left(m\cos\theta\dot{x} + m\dot{\mu}\right) \\ &= m\cos\theta\ddot{x} + m\ddot{\mu}\end{aligned} \quad (1.12)$$

$$\frac{\partial L}{\partial \mu} = mg\sin\theta - 3k\mu \quad (1.13)$$

Assembling equations (1.10) - (1.13), we have the final equations of motions

$$6m\ddot{x} + m\ddot{\mu}\cos\theta + 8kx = 0 \quad (1.14)$$

$$m\ddot{x}\cos\theta + m\ddot{\mu} - mg\sin\theta + 3k\mu = 0 \quad (1.15)$$